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COMPOSITION FACTORS OF INDECOMPOSABLE MODULES

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Abstract. Let Λ be a connected, basic finite dimensional algebra over an algebraically closed field. Our main aim is to prove that if Λ is biserial, its ordinary quiver has no loop and every indecomposable Λ-module is uniquely determined by its composition factors, then each indecomposable Λ-module is multiplicity-free.

In this article a k-algebra Λ means a finite dimensional k-algebra, where k is a fixed algebraically closed field. Denote by Λ-mod the category of all finitely generated left Λ-modules and by Λ-ind the full subcategory of Λ-mod defined by one representative of each isomorphism class of indecomposable Λ-modules. All modules and maps are in Λ-mod. A module M is called *multiplicity-free* if its composition factors are pairwise non-isomorphic.

We say that Λ-ind is *determined by composition factors* (*multiplicity-free*) if every module in Λ-ind is uniquely determined by its composition factors (respectively, multiplicity-free). For example, it is well-known that an algebra Λ is semisimple if and only if each of its modules is uniquely determined by composition factors. Though the composition factors do not determine arbitrary modules over non-semisimple algebras, it is an interesting question to know under which conditions it has the property that Λ-*ind is determined by composition factors.* In [5], Auslander and Reiten obtained sufficient conditions for Λ-ind to be determined by composition factors. Besides this question it is also an interesting question to know when an algebra has the property that Λ-*ind is multiplicity free*, and so find out if there is any relationship between those properties. It is well-known, for example, that if Λ is a representation-finite hereditary k-algebra then Λ-ind is determined by composition factors (see [2]), although it may not hold true in general that "Λ-ind is multiplicity-free". It is not difficult to show that, among the representation-finite hereditary k-algebras, the ones of the form $\Lambda = kQ$, with Λ-ind multiplicity-free, coincide with those whose quiver Q is of type A_n.

According to [12], if Λ is a k-algebra such that Λ-ind is multiplicity-free, then Λ is a representation-finite biserial and Schurian algebra. Recall that *an algebra Λ is biserial if the radical of any indecomposable non-uniserial projective, left or right, Λ-module is a sum of two uniserial submodules whose intersection is simple or zero.* Well-known examples of biserial algebras are Nakayama algebras and iterated tilted algebras of type A_n (see [1] for the definition).

With respect to a relationship between the properties Λ-*ind being determined by composition factors* and Λ-*ind being multiplicity-free*, the results of Pogorzaly and Skowroński (see [12], th.1) together with results of Skowroński and Waschbüsch (see

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[14], cor. of th.1) show that the conditions (i) Λ *is biserial and* Λ*-ind is determined by composition factors* and (ii) Λ*-ind is multiplicity-free* are equivalent, if Λ is assumed to be a factor of a hereditary algebra. (Recall that a k-algebra Λ is a factor of a hereditary algebra if and only if the corresponding ordinary quiver has neither loops nor oriented cycles.)

This leads us to the following questions: for which k-algebras does the implication (i) ⇒ (ii) or (ii) ⇒ (i) hold? Can these hold even if Λ is not a factor of a hereditary algebra? With respect to the implication (ii) implies (i) the answer is not always. We shall exhibit, in section 3, an example where Λ is a biserial k-algebra, not Nakayama, such that (ii) is satisfied but (i) is not. (It is very simple to find a Nakayama algebra with such properties).

The answer to the implication (i) implies (ii) is the main aim of this article, and it is given by the following theorem.

Theorem 6.2. *Let Λ be a connected, basic finite dimensional algebra over an algebraically closed field, whose ordinary quiver has no loop. Assume that Λ-ind is determined by composition factors. Then Λ-ind is multiplicity-free.*

In order to formulate the main result we used the diagramatic structure of the representation-finite biserial k-algebras Λ given by Skowroński and Pogorzaly (in [12], th.2). We carry this structure over to a certain "local" structure of some indecomposable modules, in the case that the ordinary quiver Q of Λ has no loop and all oriented cycles in kQ are in the relation set; and this allows us to prove an interesting result which is useful for the proof of theorem 6.2 , which is the following theorem.

Theorem 6.1. *Let* $\Lambda = k(Q_{\Lambda}, R_{\Lambda})$ *be a representation-finite biserial bound quiver algebra. Suppose that the quiver* Q_{Λ} *has no loop and that all oriented cycles of* kQ_{Λ} *are in* R_{Λ}. *If* M *is a multiplicity-free indecomposable* Λ*-module, then either* M *is projective-injective non-uniserial or* M *is of type* A_n.

As a consequence of theorems 6.1 and 6.2 we have that if Λ is a biserial k-algebra whose ordinary quiver has no loop, and such that Λ-ind is determined by composition factors, then there are two types of indecomposable modules: the projective-injective non-uniserial modules and the modules of type A_n.

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1. Preliminaries

Given a k-algebra Λ and a module M, we denote by $K_0(\Lambda)$ the Grothendieck group of Λ and by $[M]$ the image of M in $K_0(\Lambda)$. We will identify a k-algebra Λ with its associated bound quiver algebra $k(Q_\Lambda, R_\Lambda)$ (see [8], [9]), where Q_Λ denotes the ordinary quiver of Λ. Consequently we will identify the category Λ-mod with the category (Q_Λ, R_Λ)-mod of the finite k-representations of (Q_Λ, R_Λ) ([8], [9]). This enables us to denote by P_a (I_a) the indecomposable projective (injective) module which corresponds to the vertex a of Q_Λ, and by S_a the simple module associated to the vertex a. If (Q, R) is a quiver with relations, we denote by Q^{op} the quiver obtained from Q by reversing arrows and by γ^* the arrows of Q^{op} to which correspond the arrows γ of Q. So (Q^{op}, R^{op}) is the opposite quiver with relations of (Q, R), where $R^{op} = \{\rho^* \in kQ^{op} : \rho \in R\}$. To each module M

we associate a subquiver of Q_Λ, which is denoted by Q_M, as follows. The set of vertices of Q_M is the set of the vertices x of Q_Λ for which $M(x) \neq (0)$, and its set of arrows is formed by the arrows α of Q_Λ for which $M(\alpha) \neq 0$. Sometimes we refer to the set of vertices of the quiver of M as the *support* of M and denote it by $supp\, M$. It is easy to verify that if M is indecomposable, then Q_M is connected. We say that a module M is of type A_n if the underlying graph of the quiver Q_M is a diagram A_n and if $M(x) \cong k$, for each $x \in supp\, M$, and the k-morphism $M(\alpha)$ is a scalar multiple of the identity, for each arrow α of Q_M. For an algebra Λ and any Λ-module M we will denote by I_M the Λ-injective envelope of M, by $topM$ the top of M, by $socM$ the socle of M, by $radM$ the radical of M, and by $l(M)$ the length of M. Finally, by D we denote the duality $Hom_k(-,k)$, by τ we denote the Auslander-Reiten operator which maps the non-projective indecomposables one-to-one to the non-injective indecomposables (it has a natural inverse τ^{-1}), and by Γ_Λ the Auslander-Reiten quiver of Λ (see [3], [4], or [9]).

Remark 1.1. Let Λ be a k-algebra such that Λ-ind is determined by composition factors. Then Λ is representation-finite. This is an immediate consequence of BT II (see [7], [11] or [13]). It is a nice problem to prove this without using BT II. It follows from this that the ordinary quiver of Λ has neither multiple arrows nor diagrams $\tilde{A_n}$ without relations, where $n \geq 3$.

2. Nakayama algebras

In this section we consider our questions in the easy case of Nakayama algebras. Recall that for a Nakayama algebra Λ, by definition, all indecomposable projective and injective Λ-modules are uniserial. In particular, in this section the algebras mentioned are Artin algebras (that is, an artinian ring Λ with 1 such that its centre contains an artinian subring over which Λ is finitely generated as a module). We refer to [2], [6] and [8] for well known facts that we shall use here freely. Of course, as Λ is assumed connected, we assume that all simple Λ-modules lie in one τ-orbit. Thus, we can also order the representatives $P_1, P_2, ..., P_n$ of all indecomposable projective modules according to the *Kupisch series* (see [6]), that is, in such a way that $top\, P_{i+1} \cong \tau^{-1}(top\, P_i)$ for $i = 1, 2, ..., n-1$ and, if $l(P_1) \neq 1$, $top\, P_1 \cong \tau^{-1}(top\, P_n)$.

Proposition 2.1. *Let Λ be a connected, basic Nakayama algebra and n be the rank of the Grothendieck group of Λ. If $n = 1$ or $n \geq 2$ and there is a simple projective Λ-module, then Λ-ind is always determined by composition factors. If, on the contrary, $n \geq 2$ and $l(P) \geq 2$ for all indecomposable projective Λ-module, then Λ-ind is determined by composition factors if and only if $l(P) \leq n$, for all indecomposable projective P, and there is at most one of them of length n.*

Proof. The first of these assertions is an easy consequence of well known properties related to the Kupisch series of Λ, and the same is the case for the "only if" part of the last assertion.

Let us assume now that $l(M) \leq n$, for all M in Λ-ind. If M is not projective, then $l(M) < n$ and, since it is determined by its top and length, M is uniquely determined by its composition factors. But this is clearly true for any indecomposable projective Λ-module P with $l(P) < n$. Hence the result follows. □

When Λ is a Nakayama algebra, we also establish necessary and sufficient conditions for Λ-ind to be multiplicity-free.

Proposition 2.2. *Let Λ be a connected, basic Nakayama algebra and n be the rank of the Grothendieck group of Λ. Then Λ-ind is multiplicity-free if and only if every projective indecomposable Λ-module has length at most n.*

Proof. This is an immediate consequence of properties related to the Kupisch series and the fact that every indecomposable Λ-module is local. $\square$

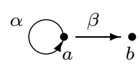
3. Some conditions for Λ-ind to be determined by composition factors

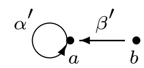
In this section we will establish some necessary conditions for Λ-ind to be determined by composition factors, in the case that Λ is a bound quiver algebra.

Let $\Lambda = k(Q, R)$ be a bound quiver algebra, where k is a fixed field. Ringel, in [13], describes some algorithms that give a way to control the category Λ-mod when, for example, we delete certain vertices or arrows of Q. It allows us to analyse Λ-mod by simpler quivers whose associated bound quiver algebra may have some suitable properties. We will use two of those algorithms, namely *AL.1: deleting for vertices* and *AL.2: deleting for arrows* (see [13]). These processes allow us to obtain a simpler quiver with relations (Q', R'), such that (Q', R')-mod is a full subcategory of (Q, R)-mod. Furthermore, they can be applied repeatedly and alternately any given number of times.

Proposition 3.1. *Let Λ be a connected, basic k-algebra. If Λ-ind is determined by composition factors, then the quiver with relations $(Q_{\Lambda}, R_{\Lambda})$ such that $\Lambda = k(Q_{\Lambda}, R_{\Lambda})$ satisfies the following conditions:*

(i) Q_{Λ} has no multiple arrow or subquivers $\tilde{A}_n$ without relations, where $n \geq 3$;

(ii) Q_{Λ} has no subquiver Q': 

(or 

), with $\beta\alpha \notin R_{\Lambda}$ (or $\alpha'\beta' \notin R_{\Lambda}$);

(iii) kQ_{Λ} has no oriented cycle $\mu = \alpha_r\alpha_{r-1}...\alpha_2\alpha_1$, with $r \geq 2$, $\alpha_i \neq \alpha_j$ if $i \neq j$, such that $\mu \notin R_{\Lambda}$ and every subpath of μ is not a summand of a generator of R_{Λ}.

Proof. Since Λ-ind is determined by composition factors, by remark 1.1, it follows that (i) is satisfied.

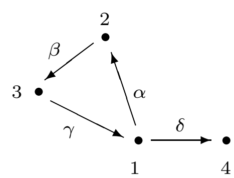
(ii) Suppose that Q_{Λ} contains a subquiver Q'. We can reduce to the case that $Q_{\Lambda} = Q'$. In fact, if Q' is a proper subquiver of Q_{Λ}, deleting the vertices $x \notin \{a, b\}$ and deleting the arrows $\gamma \notin \{\alpha, \beta\}$ (or $\gamma \notin \{\alpha', \beta'\}$), applying AL.1 and AL.2 repeatedly and alternately, we obtain the algebra $\Lambda' = k(Q', R')$, where $\beta\alpha \notin R'$ (or $\alpha'\beta' \notin R'$), whose (Q', R')-mod is a full subcategory of Λ-mod. We will write the proof of the first case only, since the other one follows similarly.

Since R_{Λ} is an admissible ideal of kQ_{Λ} and $\beta\alpha \notin R_{\Lambda}$, there are r and s in $\mathbb{Z}$ greater than or equal to 2 such that α^r and $\beta\alpha^s$ are in R_{Λ}. Without loss of generality we can suppose that $r = s = 2$. Let us consider the indecomposable projective $P_a = \Lambda(e_a + R)$, where e_a is the trivial path of kQ_{Λ} with origin at a. Then $rad\, P_a = \Lambda(\alpha + R) \oplus \Lambda(\beta + R)$ with $\Lambda(\beta + R) = k(\beta + R) \cong S_b$. Therefore $[P_a] = 2\,[S_a] + 2\,[S_b]$ in $K_0(\Lambda)$, and from this it follows that the module $M = P_a/rad^2 P_a$ is indecomposable and such that $[M] = 2\,[S_a] + [S_b]$ and $soc\, M = S_a \oplus S_b$. Let us consider the indecomposable injective $I_b = DP_b^{op}$. Since $\Lambda^{op} = k(Q^{op}, R^{op})$, it is easy to see that $[I_b] = [DP_b^{op}] = 2\,[S_a] + [S_b]$ in $K_0(\Lambda^{op})$ and $soc\, I_b = S_b$. Hence we obtain two indecomposable modules, which are the modules M and I_b, having the same composition factors, but not isomorphic, which contradicts the assumption.

(iii) Suppose that there exists in kQ_{Λ} an oriented cycle μ as in (iii). Denote by Q'' the subquiver of Q_{Λ} which is the cycle μ. Applying AL.1 and AL.2 to delete conveniently certain vertices and arrows, we obtain an algebra $\Lambda' = k(Q'', R'')$ such that $u \notin R''$ and whose (Q'', R'')-mod is a full subcategory of Λ-mod. But $\Lambda' = k(Q'', R'')$ is a connected Nakayama algebra such that the rank of $K_0(\Lambda')$ is $r \geq 2$. Since $\mu \notin R''$, it follows that there is an indecomposable projective Λ'-module P with $l(P) > r$ (for example, $P = P_{o(\alpha_1)}$). So, by proposition 2.1, Λ'-ind is not determined by composition factors, and as a consequence Λ-ind is not determined by composition factors, which is a contradiction to the assumption on Λ-ind. □

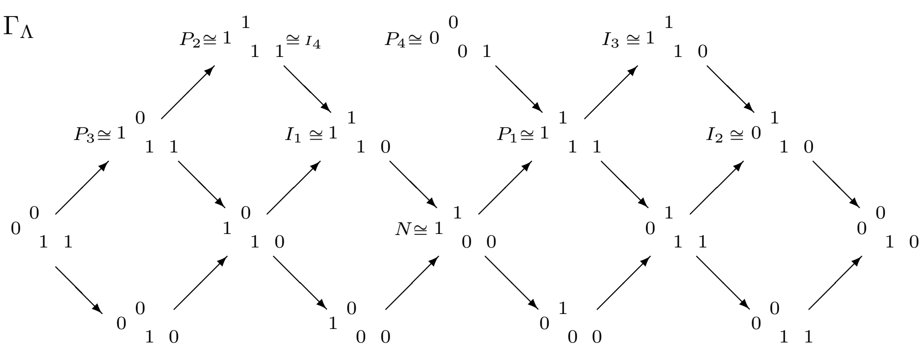
To conclude this section, we present an example of an algebra Λ whose Λ-ind is multiplicity-free but is not determined by composition factors, which shows that the property (ii) does not imply (i).

Example 3.1. Let $\Lambda = k(Q, R)$, where Q is the quiver below and R is given by $\gamma\beta\alpha = 0$ and $\alpha\gamma = 0$.



Λ is a representation-finite biserial algebra, where $dim_k\Lambda = 12$, such that all indecomposable projective modules, except P_1, and all indecomposable injective modules are uniserial (P_1 is not uniserial, since $rad\, P_1 = P_4 \oplus N$, where $N \cong P_2/rad^2\, P_2$). The Auslander-Reiten quiver of Λ, illustrated below, shows that Λ-ind is multiplicity-free. But Λ-ind is not determined by composition factors, since, for example, P_1 and P_2 are nonisomorphic modules having the same composition factors.

Γ_Λ



4. Indecomposable extensions of a simple module

In [12], Pogorzaly and Skowroński settled the diagrammatic structure of the representation-finite biserial algebras, through the following theorem.

Theorem 4.1. (see [12], th.2) *Any representation-finite biserial k-algebra Λ is isomorphic to a bound quiver algebra $k(Q_\Lambda, R_\Lambda)$, where (Q_Λ, R_Λ) satisfies the following conditions, denoted by (SP):*

(1) The number of arrows starting (ending) at any fixed vertex of Q_Λ is at most two.

(2) For any arrow α of Q_Λ there is at most one arrow β and at most one arrow γ such that $\beta\alpha$ and $\alpha\gamma$ are not in R_Λ.

(3) There is an upper bound for the length of paths in Q_Λ which are not in R_Λ.

(4) R_Λ is generated by paths and by the differences of pairs of parallel paths of Q_Λ.

From now on, when we refer to a representation-finite biserial k-algebra Λ, (Q_Λ, R_Λ) will denote a fixed bound quiver satisfying the conditions (SP) of the above theorem.

Remark 4.1. In a bound quiver (Q, R) which satisfies the above conditions (SP) it is easy to see that the following conditions are verified. **(4.1.a)** A pair of parallel paths of kQ, which are not in R, does not contain common arrows, and their vertices, except the origin and the end, are vertices in just one of these paths. **(4.1.b)** Given a path $\mu \in kQ - R$, no proper subpath of μ is a summand of a generator of R. **(4.1.c)** Given an arrow $\alpha\colon i \to j$ of Q, then the conditions (SP)1, 2, 3 determine the unique path not in R starting in α, which is the maximal path among all paths μ_α which are not in R and starting in α. We denote this path by w_α. Moreover, if Q does not contain loops or subpaths which are oriented cycles without repeated arrows, then the arrows of w_α are all distinct and each vertex of w_α, except the origin and the end, belongs to exactly two of these arrows.

To simplify the reading, we introduce some notations. Given a path w of Q, we denote by w_1 the set of the arrows in w and by w_0 the set of the vertices in w. The expression "*a path u_α*" means a path u starting in the arrow α. Moreover, we denote by $\hat{u_\alpha}$ the oriented cycle of minimal length among all the oriented cycles (if they exist), starting in α, which are not in R.

Using these notations, we state (without proof) the following technical lemma.

Lemma 4.2. *Let $k(Q, R)$ be a bound quiver algebra, where (Q, R) satisfies the conditions (SP) of theorem 4.1 and Q contains neither loops nor double arrows. Let i be a fixed vertex of Q.*

(a) *Suppose that i is the origin of exactly one arrow $\alpha : i \to j$ (notation $i = o(\alpha)$).*
- (a.1) *If the end of w_α (notation $e(w_\alpha)$) is i, then $w_\alpha = (\hat{u_\alpha})^r$, for some $r \geq 1$.*
- (a.2) *If $e(w_\alpha) \neq i$, then either (i) $w_\alpha = u'_\alpha(\hat{u_\alpha})^r$, for some $r \geq 1$, where u'_α is a proper subpath of $\hat{u_\alpha}$, or (ii). w_α does not contain oriented cycles starting at i;*

(b) *Suppose that i is the origin of two distinct arrows $\alpha : i \to j_1$ e $\beta : i \to j_2$.*
- (b.1) *If there are two oriented cycles, one starting in α and the other starting in β, then $w_\alpha = \hat{u_\alpha}$ and $w_\beta = \hat{u_\beta}$, and such oriented cycles are parallel paths (so $w_\alpha - w_\beta \in R$).*
- (b.2) *If $e(w_\alpha) = i$ and $e(w_\beta) \neq i$, then $w_\alpha = (\hat{u_\alpha})^r$, for some $r \geq 1$, w_β does not contain oriented cycles starting at i, $(w_\alpha)_0 \cap (w_\beta)_0 = i$ and $(w_\alpha)_1 \cap (w_\beta)_1 = \emptyset$.*
- (b.3) *If $e(w_\alpha) = e(w_\beta) \neq i$, then either (i) one of them, say w_α, contains an oriented cycle starting in α, and, in this case, $w_\alpha = w_\beta \hat{u_\alpha}$ and $\hat{u_\alpha}$ does not contain the arrow β, or (ii) w_α and w_β are parallel (so, $w_\alpha - w_\beta \in R$).*
- (b.4) *If $e(w_\alpha) \neq e(w_\beta)$, with $i \in \{e(w_\alpha), e(w_\beta)\}$, then either (i) $w_\alpha = u'_\alpha(\hat{u_\alpha})^r$, for some $r \geq 1$, where u'_α is a proper subpath of $\hat{u_\alpha}$, w_β contains no*

oriented cycle starting in β, $(w_\alpha)_0 \cap (w_\beta)_0 = i$ and $(w_\alpha)_1 \cap (w_\beta)_1 = \emptyset$, or (ii) $w_\alpha = u_\beta \hat{u_\alpha}$ and $w_\beta = \gamma u_\beta$, where u_β is a proper subpath of w_β starting at $e(w_\alpha)$, γ is a subpath of w_β starting at $e(w_\alpha)$ and such that $(\hat{u_\alpha})_0 \cap (w_\beta)_0 = i$, and $(\hat{u_\alpha})_1 \cap (w_\beta)_1 = \emptyset$ or (iii) w_α and w_β contain no oriented cycle starting at i, and i is the unique common vertex among them and they do not have common arrows.

The structure given by theorem 4.1, and the description in lemma 4.2 and proposition 3.1, allow us initially to prove the following theorem, where we shall use the notations above.

Theorem 4.3. *Let Λ be a biserial k-algebra whose ordinary quiver Q_Λ does not contain any loop. Suppose that Λ-ind is determined by composition factors. Then each projective indecomposable Λ-module and each injective indecomposable Λ-module is multiplicity-free.*

Proof. Using the duality D, it is enough to show that the indecomposable projective modules are multiplicity-free. By remark 1.1, Λ is representation-finite and, being a biserial algebra, by theorem 4.1 it is isomorphic to $k(Q_\Lambda, R_\Lambda)$, where (Q_Λ, R_Λ) satisfies the conditions (SP) and the conditions of proposition 3.1.

Suppose that there is a projective indecomposable module P and a simple module S such that S is a composition factor of P with multiplicity greater than or equal to 2. So, $P \cong P_i$ for some vertex i of Q_Λ, $S \cong S_j$ for some vertex j of Q_Λ and $dim_k P(j) \geq 2$.

$1^{\underline{st}}$ **case:** $S \cong S_i$ (i.e, $i = j$). Then there exists an oriented cycle in $kQ_\Lambda - R_\Lambda$, starting at i. Depending on whether i is the origin of one arrow or of two distinct arrows, we have one of the cases (a.1), (a.2.i), (b.1), (b.2), (b.3.i), (b.4.i) or (b.4.ii) in lemma 4.2. If we have case (b.1), then the maximal paths w_α and w_β are parallel and $w_\alpha - w_\beta \in R_\Lambda$. Hence,

$$rad\, P_i = \Lambda(\alpha + R_\Lambda) + \Lambda(\beta + R_\Lambda) \quad \text{and} \quad soc\, P_i = k(w_\alpha + R_\Lambda) = k(w_\beta + R_\Lambda) \cong S_i.$$

So the indecomposable modules $M = P_i/\Lambda(\beta + R_\Lambda)$ and $N = \Lambda(\alpha + R_\Lambda)$ have the same composition factors (the simple modules S_x, where $x \in (w_\alpha)_0$), but they are not isomorphic, since $top\, M \cong S_i$ and $top\, N \cong S_{e(\alpha)}$. This is a contradiction to Λ-ind being determined by composition factors. In any of the cases (a.1), (a.2.i), (b.2), (b.3.i), (b.4.i) or (b.4.ii), using the notation of the last lemma, we can consider the oriented cycle $\hat{u_\alpha}$. By the minimality of the length of $\hat{u_\alpha}$, it contains no repeated arrows and, by remark 4.1.b, this cycle contains no subpath which is a summand of a generator of R_Λ. Then we get a contradiction to the conditions on (Q_Λ, R_Λ).

$2^{\underline{nd}}$ **case:** $S \ncong S_i$ (i.e, $i \neq j$). In this case, there is no oriented cycle in Q_Λ starting at i which is not in R_Λ. Then, from the condition (SP)4 and from $dim_k P(j) = dim_k P_i(j) \geq 2$, it follows that there is an oriented cycle $\mu \notin R_\Lambda$, starting at j, such that i is not a vertex of μ. So, relative to the paths with origin at i, we have one of the remaining cases of lemma 4.2, that is, (a.2.ii), (b.3.ii) or (b.4.ii). In case (a.2.ii), μ is a proper subpath of w_α and j is the end of two arrows γ and γ', with γ' in μ, and j is the origin of at least one arrow δ', with δ' in μ, such that $\mu\gamma \notin R_\Lambda$ and $\delta'\gamma' \in R_\Lambda$ (if there is another arrow δ in w_α, starting at j, then $\delta\mu\gamma \notin R_\Lambda$ and $\delta\gamma \in R_\Lambda$). From this, it follows that μ is the unique oriented cycle starting at j, and so $\mu = \hat{u_{\delta'}}$ (since $\delta'\gamma' \in R_\Lambda$). In case (b.3.ii), μ is a proper subpath of exactly one of the parallel paths w_α and w_β, let us say w_α, and j is not the end of any of them. Therefore, j is the end of two distinct arrows γ and γ' of w_α and the origin of two distinct arrows δ and δ' as in (a.2.ii), and hence, $\mu = \hat{u_{\delta'}}$. If we have case (b.4.ii), since $(w_\alpha)_1 \cap (w_\beta)_1 = \emptyset$ and $(w_\alpha)_0 \cap (w_\beta)_0 = \{i\}$, we infer that μ is a proper subpath of only one of them, and relative to the arrows which have vertices at j it behaves as in case (a.2.i). Hence, in any of these possibilities, as in the first case, Q_Λ contains an oriented cycle, not in R_Λ, whose arrows are all different and such that no subpath of it is a summand of a generator of R_Λ, which gives a contradiction to condition (iii) of proposition 3.1. So, any projective indecomposable Λ-module is multiplicity-free. □

Remark 4.2. In fact, the proof of theorem 4.3 follows from the fact that any oriented cycle of kQ_Λ is in R_Λ. Therefore if in this theorem we require that all oriented cycles are in the relation set instead of Λ-ind being determined by composition factors, its statement is still true.

To attain our aim, we need some preliminaries. Given a simple module S, we denote by i_S the vertex of Q_Λ which is associated to S. We start with another technical lemma.

Lemma 4.4. *Let Λ be a representation-finite biserial algebra whose ordinary quiver Q_Λ has no loop and such that all oriented cycles are in R_Λ. Let N be a Λ-module, with $l(N) \geq 2$, and let $v \in N$ be such that the simple Λ-module $S = kv \subset N$ is not a direct summand of N. Then*

(i) *There is a path*

$$\underset{y_r}{\bullet} \xrightarrow{\sigma_r} \underset{y_{r-1}}{\bullet} \longrightarrow \bullet \cdots \underset{y_2}{\bullet} \xrightarrow{\sigma_2} \underset{y_1}{\bullet} \xrightarrow{\sigma_1} \underset{y_0 = i_S}{\bullet}, \quad r \geq 1,$$

and a family $(v_i)_i \in \bigoplus_{i=0}^r N(y_i)$, with $v_0 = v$, such that $N(y_i) = kv_i \oplus N'(y_i)$, $N(\sigma_i)(v_i) = v_{i-1}$ and $N(\sigma_i)(N'(y_i)) \subset N'(y_{i-1})$, for $1 \leq i \leq r$, and $Im\, N(\delta) \subset N'(y_r)$, for any arrow δ ending at y_r.

(ii) *If there are two arrows $\sigma : y \to i_S$ and $\overline{\sigma} : \overline{y} \to i_S$, with $\sigma \neq \overline{\sigma}$, such that $v \in Im\, N(\sigma) \cap Im\, N(\overline{\sigma})$, then the quiver Q_N of N contains the subquiver*

$$\underset{y_r}{\bullet} \xrightarrow{\sigma_r} \underset{y_{r-1}}{\bullet} \longrightarrow \bullet \cdots \underset{y_2}{\bullet} \xrightarrow{\sigma_2} \underset{y_1}{\bullet} \xrightarrow{\sigma_1} \underset{y_0 = i_S}{\bullet} \xleftarrow{\overline{\sigma}_1} \underset{\overline{y}_1}{\bullet} \xleftarrow{\overline{\sigma}_2} \underset{\overline{y}_2}{\bullet} \cdots \bullet \longleftarrow \underset{\overline{y}_{m-1}}{\bullet} \xleftarrow{\overline{\sigma}_r} \underset{\overline{y}_m}{\bullet},$$

with $r, m \geq 1$, $\sigma_1 = \sigma$, $\overline{\sigma}_1 = \overline{\sigma}$. For this subquiver there are the families

$$(v_i)_i \in \bigoplus_{i=0}^{r} N(y_i) \text{ and } (\overline{v}_j)_j \in \bigoplus_{j=0}^{m} N(\overline{y}_j),$$

where $\overline{v}_0 = v = v_0$, such that $N(y_i) = kv_i \oplus N'(y_i)$, $N(\sigma_i)(v_i) = v_{i-1}$, $N(\sigma_i)(N'(y_i)) \subset N'(y_{i-1})$, $1 \leq i \leq r$, $N(\overline{y}_i) = k\overline{v}_i \oplus N'(\overline{y}_i)$, $N(\overline{\sigma}_i)(\overline{v}_i) = \overline{v}_{i-1}$, $N(\overline{\sigma}_i)(N'(\overline{y}_i)) \subset N'(\overline{y}_{i-1})$, $1 \leq i \leq m$. Furthermore, $Im\, N(\delta) \subset N'(y_r)$, for all arrows δ ending at y_r, and $Im\, N(\overline{\delta}) \subset N'(\overline{y}_m)$, for all arrows $\overline{\delta}$ ending at $\overline{y}_m$.

Proof. It follows, from remark 4.2, that each injective indecomposable Λ-module I is of type A_n or a projective-injective non-uniserial. Therefore, $I(x) \cong k$ for all $x \in supp\, I$, and $I(\alpha) \cong \mathbb{1}$ for every arrow α of the quiver Q_I.

We consider $N(i_S) = kv \oplus N'(i_S)$. Let $\iota \colon S \hookrightarrow N$ be the inclusion of $S = kv$ in N and $\jmath \colon S \hookrightarrow I_S$ be the injective envelope of S. Then, there exists a morphism of k-representations $\lambda = (\lambda_x)_x \colon N \to I_S$ such that $\lambda \circ \iota = \jmath$. Since $I_S(i_S) = kv$, we have that $ker \lambda_{i_S} = N'(i_S)$. So, since S is not a direct summand of N, $L = Im \lambda$ is a submodule of I_S containing S properly, and hence it is an indecomposable module with $l(L)\geq 2$. Since I_S is of type A_n or a projective-injective non-uniserial, there is a subpath μ: $y_r \overset{\sigma_r}{\to} y_{r-1}\dots y_2 \overset{\sigma_2}{\to} y_1=y \overset{\sigma_1}{\to} y_0=i_{_S}$, $r\geq 1$, in the quiver of L, such that $(Im\,\lambda_{y_i})(\sigma_i)\,\cong\,\mathbb{1}$ for $0\leq i\leq r$, and $\lambda_x=0$ for each vertex $x=o(\delta)$, for every arrow δ, where $e(\delta)=y_r$. It means that μ is the path of maximal length of Q_L ending at σ_1 with the property $L(\mu)\,\cong\,\mathbb{1}$. Moreover, for each $i=1,2,...,r$, since $\lambda_{y_i}\neq 0$, the following diagram is commutative:

$$\begin{array}{ccc} N(y_i) & \xrightarrow{\lambda_{y_i}} & I_S(y_i)\,\cong\,k \\ {\scriptstyle N(\sigma)}\big\downarrow & & \big\downarrow{\scriptstyle \cong\,1} \\ N(y_{i-1}) & \xrightarrow{\lambda_{i_{y_{i-1}}}} & I_S(y_{i-1})\,\cong\,k \end{array}$$

From the commutativity of the above diagram and from $L\,\cong\,N/ker\lambda$, we infer that we can choose a family $(v_i)_i\,\in\,\bigoplus_{i=0}^{r}N(y_i)$, where $v_{_0}\,=\,v$, such that $N(y_i)\,=\,kv_i\oplus N'(y_i)$, where $N'(y_i)\,=\,ker\lambda_{y_{_i}}$, and such that $N(\sigma_i)(v_i)\,=\,v_{i-1}$ and $N(\sigma_i)(N'(y_i))\,\subset\,N'(y_{i-1})$, for $1\leq i\leq r$. Moreover, it follows easily that $Im\,N(\delta)\,\subset\,ker\,\lambda_{y_r}=N'(y_r)$, for each δ of Q_N, where $e(\delta)=y_r$ (since $\lambda_x=0$, where $x=o(\delta)$). So, (i) is proved.

Let $\sigma\,:\,y\,\to\,i_{_S}$ and $\overline{\sigma}\,:\,\overline{y}\,\to\,i_{_S}$ be the two arrows of $Q_{_\Lambda}$, with $\sigma\neq\overline{\sigma}$, such that $v\in Im\,N(\sigma)\cap Im\,N(\overline{\sigma})$. Since $Q_{_\Lambda}$ does not have loops or double arrows, then $\overline{y}\notin\{y,i_{_S}\}$. In this case, I_S is a non-uniserial module, and it follows from this that $l(L)\geq 3$ and that λ_y and $\lambda_{\overline{y}}$ are not zero. So, L is a non-uniserial submodule of I_S, whose quiver is $y_r \overset{\sigma_r}{\to} y_{r-1}\dots y_2 \overset{\sigma_2}{\to} y_1=y \overset{\sigma_1}{\to} y_{_0}=i_{_S} \overset{\overline{\sigma}_1}{\leftarrow} \overline{y}_1=\overline{y} \overset{\overline{\sigma}_2}{\leftarrow} \overline{y}_2\dots\overline{y}_{m-1} \overset{\overline{\sigma}_m}{\leftarrow} \overline{y}_m$, where $r,m\geq 1$, $\sigma_1=\sigma$, $\overline{\sigma}_1=\overline{\sigma}$. (We observe that, if $\overline{y}_m=y_r$, then I_S is an injective-projective non-uniserial and λ is an epimorphism). The rest of the proof follows using arguments as in part (i) on each of the two paths ending at $i_{_S}$, which make Q_L. □

Remark 4.3. Under the hypothesis of lemma 4.4, if there are two different arrows $\sigma\colon y\to i_{_S}$ and $\overline{\sigma}\colon\overline{y}\to i_{_S}$ of $Q_{_\Lambda}$ such that $v\in Im\,N(\sigma)$ but $v\notin Im\,N(\overline{\sigma})$, then, by a conveniently chosen basis for $N(\overline{y})$, we infer that $Im\,N(\overline{\sigma})\subset N'(i_{_S})$. Furthermore, using (SP)2, we can describe the morphism $N(\alpha)$, for each arrow α different from σ_i, for $1\leq i\leq r$, having one of its vertices at y_i, $0\leq i\leq r$. If $\alpha\neq\sigma_i$, $1\leq i\leq r-1$, with $o(\alpha)\in\{y_i,i_{_S}\}$, then $N(\alpha)(v_i)=0$ for $1\leq i\leq r$; if $\alpha\neq\sigma_{i+1}$, $1\leq i\leq r-1$, with $e(\alpha)=y_i$, then $Im\,N(\alpha)\subset N'(y_i)$. Even in the case that $v\in Im\,N(\sigma)\cap Im\,N(\overline{\sigma})$, we can also describe similarly the morphisms $N(\gamma)$, for each $\gamma\neq\overline{\sigma}_j$, $1\leq j\leq m$, and having one of its vertices at $\overline{y}_j$, for $1\leq j\leq m-1$.

The last lemma and remark 4.3 allow us to determine if a given module M is indecomposable or not through of the number of arrows ending at the vertices of the quiver of M, which are associated to their simple submodules.

Theorem 4.5. *Let $\Lambda=k(Q_{_\Lambda},\,R_{_\Lambda})$ be a representation-finite biserial algebra whose ordinary quiver $Q_{_\Lambda}$ has no loop and such that all oriented cycles of $Q_{_\Lambda}$ are in $R_{_\Lambda}$. Let M be a Λ-module such that $M=N+X$, with $l(N)\,,l(X)\geq 2$, and suppose that there exists $0\neq v\in M$ such that $N\cap X=kv=S$. If there is an arrow σ ending at $i_{_S}$ such that $v\in Im\,N(\sigma)\cap Im\,X(\sigma)$, then M is decomposable.*

Proof. Since $S = kv \subset M$, it follows that $M(\alpha)(v) = 0$ for each arrow α where $o(\alpha) = i_S$. On the other hand, setting $N(i_S) = kv \oplus N'(i_S)$ and $X(i_S) = kv \oplus X'(i_S)$, we can rewrite $M = (M(x)_x, M(\alpha)_\alpha)$ in the following way:

$$M(x) = \begin{cases} N(x) \oplus X(x) & \text{if } x \neq i_S, \\ kv \oplus N'(i_S) \oplus X'(i_S) & \text{if } x = i_S, \end{cases}$$

$$M(\alpha) = \begin{cases} N(\alpha) \oplus X(\alpha) & \text{if } i_S \notin \{o(\alpha), e(\alpha)\}, \\ N(\alpha) + X(\alpha) & \text{if } e(\alpha) = i_S, \\ N'(\alpha) \oplus X'(\alpha) & \text{if } o(\alpha) = i_S, \end{cases}$$

where $X'(\alpha)$ and $N'(\alpha)$ denote, respectively, the restriction of $X(\alpha)$ to the subspace $X(o(\alpha))$ and of $N(\alpha)$ to the subspace $N(o(\alpha))$. It is clear that S is neither a direct summand of N nor of X. We consider several cases.

Let us assume first that, besides the arrow $\sigma\colon y \to i_S$, there is in Q_M another arrow $\overline{\sigma}\colon \overline{y} \to i_S$, where $\overline{\sigma} \neq \sigma$ (hence $y \neq \overline{y}$) and that $v \in Im\, N(\overline{\sigma}) \cap Im\, X(\overline{\sigma})$. Applying lemma 4.4(ii) to N and X, we see that there is a subquiver μ_N of Q_N, where $\mu_N\colon\ y_r \overset{\sigma_r}{\to} \dots \to y_s \overset{\sigma_s}{\to} \dots \to y_2 \overset{\sigma_2}{\to} y_1 = y \overset{\sigma_1}{\to} y_0 = i_S \overset{\overline{\sigma}_1}{\leftarrow} \overline{y}_1 = \overline{y} \overset{\overline{\sigma}_2}{\leftarrow} \overline{y}_2 \leftarrow \dots \overset{\overline{\sigma}_m}{\leftarrow} \overline{y}_m$, and there is a subquiver μ_X of Q_X, where $\mu_X\colon\ y_s \overset{\sigma_s}{\to} \dots \to y_2 \overset{\sigma_2}{\to} y_1 = y \overset{\sigma_1}{\to} y_0 = i_S \overset{\overline{\sigma}_1}{\leftarrow} \overline{y}_1 = \overline{y} \overset{\overline{\sigma}_2}{\leftarrow} \overline{y}_2 \leftarrow \dots \leftarrow \overline{y}_{q-1} \overset{\overline{\sigma}_q}{\leftarrow} \overline{y}_q$, where $1 \leq s \leq r$, and $q,\, m \geq 1$, $\sigma_1 = \sigma$, $\overline{\sigma}_1 = \overline{\sigma}$. For these subquivers, also by lemma 4.4(ii), there are families $(v_i)_i \in \bigoplus_{i=0}^r N(y_i)$, $(\overline{v}_i)_i \in \bigoplus_{i=0}^m N(\overline{y}_i)$, and $(w_j)_j \in \bigoplus_{j=0}^s X(y_j)$ and $(\overline{w}_j)_j \in \bigoplus_{j=0}^q, X(\overline{y}_j)$, where $v_0 = w_0 = \overline{v}_0 = \overline{w}_0 = v$, such that $N(y_i) = kv_i \oplus N'(y_i)$, for $0 \leq i \leq r$, $N(\overline{y}_i) = k\overline{v}_i \oplus N'(\overline{y}_i)$, for $0 \leq i \leq m$, and $X(y_j) = kw_j \oplus X'(y_j)$, for $0 \leq j \leq s$ and $X(\overline{y}_j) = k\overline{w}_j \oplus X'(\overline{y}_j)$, for $0 \leq j \leq q$. Moreover, the morphisms $X(\alpha)$ and $N(\alpha)$ are as in lemma 4.4 and in remark 4.3, for the arrows α starting (ending) at y_i and $\overline{y}_j$.

Since all the paths of $kQ_\Lambda - R_\Lambda$ ending in $\overline{\sigma}$ are ordered by their lengths, it follows that $q \leq m$ or $m \leq q$. Suppose now that $1 \leq m \leq q$. Using the previous notation, let $L = (L(x)_x, L(\alpha)_\alpha)$ be defined by

$$L(x) = \begin{cases} kv_i \oplus N'(y_i) \oplus X'(y_i) & \text{if } x = y_i, \text{ for } 0 \leq i \leq s, \\ kv_i \oplus N'(y_i) \oplus X(y_i) & \text{if } x = y_i, \text{ for } s+1 \leq i \leq r, \\ N'(\overline{y}_j) \oplus k\overline{w}_j \oplus X'(\overline{y}_j) & \text{if } x = \overline{y}_j, \text{ for } 1 \leq j \leq m, \\ N(\overline{y}_j) \oplus k\overline{w}_j \oplus X'(\overline{y}_j) & \text{if } x = \overline{y}_j, \text{ for } m+1 \leq j \leq q, \\ N(x) \oplus X(x) & \text{for the other vertices,} \end{cases}$$

$$L(\alpha) = \begin{cases} N(\sigma_i) \oplus X'(\sigma_i) & \text{if } \alpha = \sigma_i, \text{ for } 1 \leq i \leq s, \\ N'(\overline{\sigma}_i) \oplus X(\overline{\sigma}_i) & \text{if } \alpha = \overline{\sigma}_i, \text{ for } 1 \leq i \leq m, \\ N(\alpha) \oplus X(\alpha) & \text{for the other arrows.} \end{cases}$$

It is easy to verify that L is a representation of $(Q_\Lambda,\, R_\Lambda)$. Let $\imath = (\imath_x)_x\colon L \to M$ be a family where, for each x, $\imath_x\colon L(x) \to M(x)$ is the natural inclusion from the space $L(x)$ to the space $M(x)$. It is also easy to prove that it admits a left inverse morphism $p = (p_x)_x\colon M \to L$. The components p_x of p are the following. For $1 \leq i \leq s$, $p_{y_i}\colon kv_i \oplus N'(y_i) \oplus kw_i \oplus X'(y_i) \to kv_i \oplus N'(y_i) \oplus X'(y_i)$ is defined by $p_{y_i}(av_i + n' + bw_i + x') = (a+b)v_i + n' + x'$, with $a,\, b \in k$, $x' \in X'(y_i)$ and $n' \in N'(y_i)$; for $1 \leq i \leq m$, $p_{\overline{y}_i}\colon k\overline{v}_i \oplus N'(\overline{y}_i) \oplus k\overline{w}_i \oplus X'(\overline{y}_i) \to N'(\overline{y}_i) \oplus k\overline{w}_i \oplus X'(\overline{y}_i)$ is defined by $p_{\overline{y}_i}(a\overline{v}_i + n' + b\overline{w}_i + x') = (a+b)\overline{w}_i + n' + x'$, with $a,\, b \in k$, $n' \in N'(\overline{y}_i)$ and $x' \in X'(\overline{y}_i)$ and, for the remaining vertices x, $p_x : N(x) \oplus X(x) \to N(x) \oplus X(x)$ is the identity. We leave the verification to the reader.

For the remaining cases (that is, either $v \in Im\, N(\overline{\sigma}) \cap Im\, X(\overline{\sigma})$ and $q \leq m$, or $v \notin Im\, N(\overline{\sigma}) \cap Im\, X(\overline{\sigma})$, or σ is the unique arrow of $Q_{_M}$ ending at $i_{_S}$ such that $v \in Im\, N(\sigma) \cap Im\, X(\sigma)$), by the application of lemma 4.4 and remark 4.3 and a slight modification of the above arguments, it is not difficult to verify that $M = N \oplus X/kv$. So the theorem is proved. □

Corollary 4.6. *Let $\Lambda = k(Q_{_\Lambda}, R_{_\Lambda})$ be a representation-finite biserial k-algebra whose ordinary quiver $Q_{_\Lambda}$ does not have loops and such that all oriented cycles of $Q_{_\Lambda}$ are in $R_{_\Lambda}$. Let M be a Λ-module such that M=N+X, where $l(N), l(X) \geq 2$ and $N \cap X = S = kv$, for some $0 \neq v \in M$. If M is indecomposable, then there is a unique arrow $\sigma : y \to i_{_S}$ such that $v \in Im\, N(\sigma)$, and a unique arrow $\overline{\sigma} : \overline{y} \to i_{_S}$, where $\sigma \neq \overline{\sigma}$, such that $v \in Im\, X(\overline{\sigma})$.*

Proof. The proof follows immediately from the fact that M is indecomposable and from lemma 4.4 and theorem 4.5. □

Theorem 4.7. *Let $\Lambda = k(Q_{_\Lambda}, R_{_\Lambda})$ be a representation-finite biserial algebra whose ordinary quiver $Q_{_\Lambda}$ has no loop and such that all oriented cycles of $kQ_{_\Lambda}$ are in $R_{_\Lambda}$. Let M be a Λ-module and S be a simple Λ-module such that $S \subset M$ and $M/S = M_1 \oplus M_2 \oplus \ldots \oplus M_t$, where $t \geq 1$ and each M_j is indecomposable, for $j = 1, 2, \ldots, t$. If M is indecomposable, then $t \leq 2$.*

Proof. Let $0 \neq v \in M$ be such that $S = kv$. We consider now the canonical epimorphism $\pi \colon M \to M/S$. For each $j = 1, 2, ..., t$, we put $N_j = \pi^{-1} M_j$, the reciprocal image of M_j under π. It is clear that $M = \sum_{j=1}^{t} N_j$ and $N_i \cap N_j = S$, if $i \neq j$. Moreover, $N_j/S = M_j$, for $j = 1, 2, ..., t$. It is clear that each N_j is an indecomposable module.

Suppose now that $t \geq 3$. If we write $M = N_1 + X$, where $X = \sum_{j=2}^{t} N_j$, it follows that $N_1 \cap X = S = kv$ and, by the corollary above, there is a unique arrow σ, ending at $i_{_S}$, such that $v \in Im\, N_1(\sigma)$, and there is a unique arrow $\overline{\sigma}$, ending at $i_{_S}$, with $\overline{\sigma} \neq \sigma$, such that $v \in Im\, X(\overline{\sigma})$. So, applying lemma 4.4(i) to each N_j, for $j = 2, 3, ..., t$, and recalling that $Im\, X(\overline{\sigma}) = \sum_{j=2}^{t} Im\, N_j(\overline{\sigma})$, we see that $v \in Im\, N_j(\overline{\sigma})$, for all $j = 2, 3, ..., t$. On the other hand, if we write $X = N_2 + (N_3 + ... + N_t)$, then from theorem 4.5 it follows that $X = X_1 \oplus X_2$, where $X_i \neq (0)$ for $i = 1, 2$, and $S \subset X_1$. Therefore, $M = N_1 + (X_1 \oplus X_2) = X_2 + (N_1 + X_1)$; since $M/S = M_1 \oplus M_2 \oplus ... \oplus M_t$, and $N_j = \pi^{-1} M_j$, for $j = 1, 2, ..., t$, then $X_2 \cap (N_1 + X_1) = (0)$. This means that $M = X_2 \oplus (N_1 + X_1)$, which is a contradiction to the assumption that M is indecomposable. □

We observe that, given a module M and a simple submodule S of M, the results which we have gotten until now rely on the existence of paths in the quiver of M ending at the vertex corresponding to S. By duality, we can get the dual results about the simple modules T which are direct summands of *top* M.

5. The multiplicity of the composition factors of some modules

We are now going to measure the multiplicity of a simple submodule S of a given module M, when the quotient M/S is of type A_n for some $n \geq 2$, and also when M/S is a direct sum of two submodules of type A_n, in case Λ is a biserial algebra whose ordinary quiver has no loop and Λ-ind is determined by composition factors. Dually, we can measure the multiplicity of a simple module T as a composition factor of M, where there is an epimorphism from M to T whose *kernel* is of type A_n or is a direct sum of two submodules of type A_n.

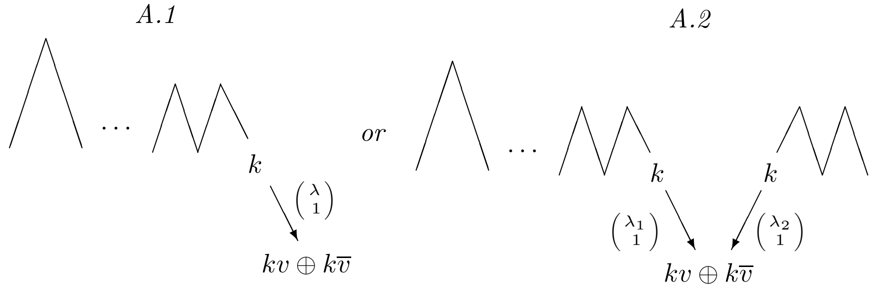
We first need to introduce some notations and conventions which make reading easier. Let Q' be a quiver whose underlying graph is a diagram A_n, for $n \geq 3$. A vertex of Q' which belongs to exactly two arrows will be called an *interior vertex* of Q'; and a vertex of Q' which belongs to exactly one arrow will be called an *extreme vertex* of Q'. So, all vertices of Q', except two of them, are interior vertices. The extreme vertices can be distinguished in the following way: if x is the origin of the unique arrow of Q' to which it belongs, we call x the *initial extreme*; otherwise, we call it the *terminal extreme*. On the other hand, the interior vertices can be divided into three classes. If x is an interior vertex such that it is the origin of the two arrows to which it belongs, we call x a *2-source*. If x is the end of the two arrows to which it belongs, we call x a *2-sink*. In the remaining case, x is called a *transit point*.

We now fix the following convention: Let M' be a module of type A_n, with $n \geq 2$. If the quiver of M' contains a proper subquiver whose orientation is not important for the context, we shall represent this subquiver by points (vertices) and by edges.

Using the notations and conventions above, we have:

Lemma 5.1. *Let $\Lambda = k(Q_\Lambda, R_\Lambda)$ be a representation-finite biserial algebra whose ordinary quiver Q_Λ has no loop. Let M be a Λ-module whose quiver is connected, and $S = kv \subset M$ be a simple submodule, where $0 \neq v \in M$. Suppose that $\overline{M} = M/S$ is of type A_n, $n \geq 2$, and that there is $\overline{v} \in \overline{M}(i_S)$ such that $\overline{S} = k\overline{v} \subset \overline{M}$. Then M is one of the following k-representation of (Q_Λ, R_Λ), relative to its support:*

A.1 *A.2*

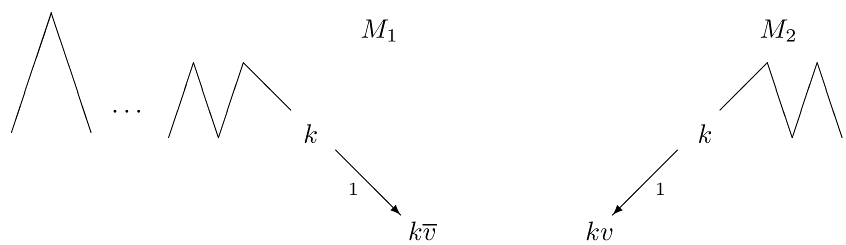


$(\lambda = 0$ *or* $\lambda \cong \mathbb{1})$ $(\lambda_i = 0,\ i = 1, 2,$ *or* $\lambda_i \cong \mathbb{1},\ i = 1, 2)$

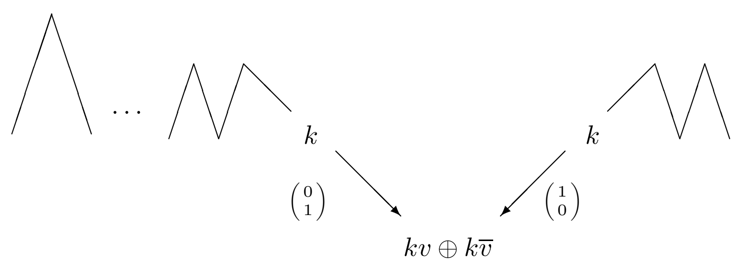
Moreover, S is a direct summand of M.

Proof. From the assumptions on $\overline{M}$, we write $M(i_S) = kv \oplus k\overline{v}$ and $M(x) = \overline{M}(x) \cong k$, for all $x \in supp\, M$, $x \neq i_S$. Since $S = kv \subset soc\, M$ and $k\overline{v} \subset \overline{M}$, it follows that $M(\delta)(v) = \overline{M}(\delta)(v) = 0$ for each arrow δ, where $o(\delta) = i_S$, and $M(\alpha) = \overline{M}(\alpha)$, for each arrow α of Q_Λ, with $i_S \notin \{o(\alpha), e(\alpha)\}$. So, the quiver of $\overline{M}$ and, consequently, the quiver of M does not contain arrows starting at i_S. Therefore, we conclude that i_S is a sink in Q_M. Since the quiver $Q_{\overline{M}}$ is obtained from the quiver Q_M by eventual exclusion of the arrows to which i_S belongs, we need to analyse the neighborhood of i_S in $Q_{\overline{M}}$.

1$^{\underline{st}}$ case: *i_S is a 2-sink in $Q_{\overline{M}}$.* Under this assumption, we infer that the quivers Q_M and $Q_{\overline{M}}$ are equal and i_S is a 2-sink of Q_M. Denote by σ_1 and σ_2, where $\sigma_1 \neq \sigma_2$, arrows of Q_M such that $e(\sigma) = i_S$ for $i = 1, 2$. Then, for $i = 1, 2$, $M(\sigma_i) \cong \begin{pmatrix} \lambda_i \\ 1 \end{pmatrix}$, where $\lambda_i = 0$ or $\lambda_i \cong \mathbb{1}$. Suppose that $\lambda_1 = 0$ and $\lambda_2 \cong \mathbb{1}$, and let M_1 and M_2 be the k-subrepresentations of M, which are represented, relative to its support, by



These k-representations are such that $k\overline{v} \subset soc\, M_1$ and $kv \subset soc\, M_2$. Consider $M_1 \oplus M_2$, which is represented by



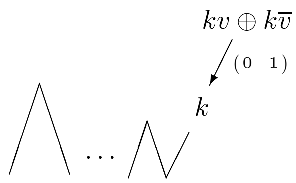
The family $f = (f_x)_x \colon M \longrightarrow M_1 \oplus M_2$, given by $f_x = \mathbb{1}_{M(x)}$ if $x \neq i_S$ and $f_{i_S}(av + b\overline{v}) = av + (b-a)\overline{v}$ for $a, b \in k$, defines an isomorphism of k-representations. Hence, $M/S \cong M_1 \oplus M_2/kv$, and it is not of type A_n. Therefore, $\lambda_i = 0$ for each $i = 1, 2$, or $\lambda_i \cong \mathbb{1}$ for each $i = 1, 2$, and M is one of the representation in A.2. In the first case, it is very simple to verify that S is a direct summand of M. In the second case, we have that the natural inclusion from kv in M is a monomorphism which admits as a left inverse morphism of k-representations the family $p = (p_x)_x \colon M \longrightarrow S = kv$, defined by by $p_x = 0$ if $x \neq i_S$ and $p_{i_S}(av + b\overline{v}) = (a-b)v$ for $a, b \in k$.

2$^{\underline{nd}}$ case: *i_S is a terminal extreme in $Q_{\overline{M}}$.* Let $\sigma \colon y \to i_S$ be the unique arrow of $Q_{\overline{M}}$ to which i_S belongs. Then $M(\sigma) = \begin{pmatrix} \lambda \\ 1 \end{pmatrix}$, where $\lambda \colon k \to kv$ is zero or is a scalar multiple of the identity. Since Q_Λ does not contain subquivers of type $\tilde{A}_m$ without relations and verifies (SP)2, and since $supp\, M = supp\, \overline{M}$, it is easy to verify that σ is the unique arrow of Q_Λ, ending at i_S, which is an arrow of Q_M. This implies that the quivers Q_M and $Q_{\overline{M}}$ coincide, and we have case A.1. By the same arguments used in the first case, we obtain that the natural inclusion from $S = kv$ to M splits, that is, S is a direct summand of M. □

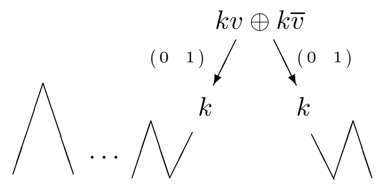
In the following lemma, we consider a k-representation M whose *top* admits a composition factor isomorphic to a direct summand of $soc\, M$.

Lemma 5.2. *Let $\Lambda = k(Q_\Lambda, R_\Lambda)$ be a representation-finite biserial algebra whose ordinary quiver has no loop. Let M be a Λ-module whose quiver Q_M is connected. Consider $v \in M$, $v \neq 0$, and $S = kv \subset M$, a simple module. Suppose that $\overline{M} = M/S$ is of type A_n, $n \geq 2$. If there is $0 \neq \overline{v} \in \overline{M}(i_S)$ such that there exists an epimorphism* *from $\overline{M}$ to $\overline{S}=k\overline{v}$, then M is one of the following k-representations, relative to its support:*

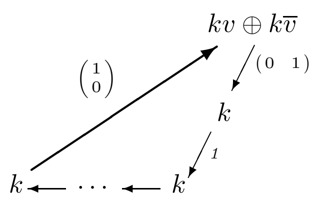
B.1



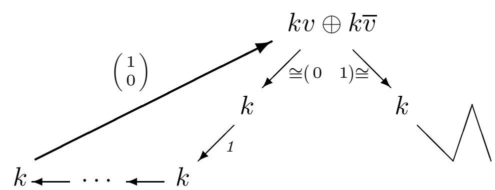
B.3



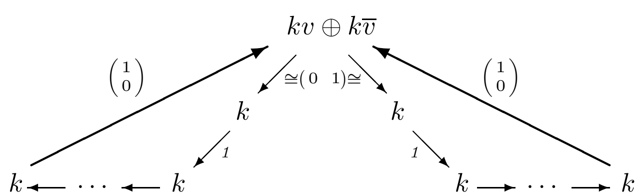
B.2



B.4



B.5



Furthermore, in cases B.1 and B.3, $S=kv$ is a direct summand of M.

Proof. As before, we can write $M(i_{_S}) = kv \oplus k\overline{v}$ and $M(x) = \overline{M}(x) \cong k$, for all vertex $x \neq i_{_S}$ of $Q_{_M}$, and $M(\alpha)=\overline{M}(\alpha)$ for all arrows α with $i_{_S} \notin \{o(\alpha), e(\alpha)\}$, and $M(\delta)(v) = 0$ for each arrow δ with $o(\delta) = i_{_S}$. From the hypothesis on $\overline{M}$ and $\overline{S}$, as in the previous lemma, it is easy to see that $i_{_S}$ is a source of $Q_{\overline{M}}$. Moreover, from the epimorphism $M \to \overline{M} \to k\overline{v}$, it follows that $k\overline{v} \cap \sum_\gamma Im\, M(\gamma) = (0)$, for each arrow γ, where $e(\gamma) = i_{_S}$. We have now some cases, according to which kind of source $i_{_S}$ is of $Q_{\overline{M}}$ and if there is an arrow ending at $i_{_S}$ in $Q_{_M}$.

1$^{\underline{st}}$ case: *$i_{_S}$ is a 2-source of $Q_{\overline{M}}$.* Consider $\delta_1 : i_{_S} \to t_1$ and $\delta_2 : i_{_S} \to t_2$, where $\delta_1 \neq \delta_2$ (hence, $t_1 \notin \{i_{_S}, t_2\}$), such that $\overline{M}(\delta_1) \cong \mathbb{1} \cong \overline{M}(\delta_2)$. So, $M(\delta_1) \cong (\,0 \quad 1\,) \cong M(\delta_2)$. According to whether $i_{_S}$ is an end point of arrows in $Q_{_M}$, we have the following subcases.

1.1: *There is no arrow in $Q_{_M}$ ending at $i_{_S}$ (i.e, $i_{_S}$ is a 2-source of $Q_{_M}$).* This condition implies that the quivers $Q_{_M}$ and $Q_{\overline{M}}$ coincide, and so M is the k-representation in B.3.

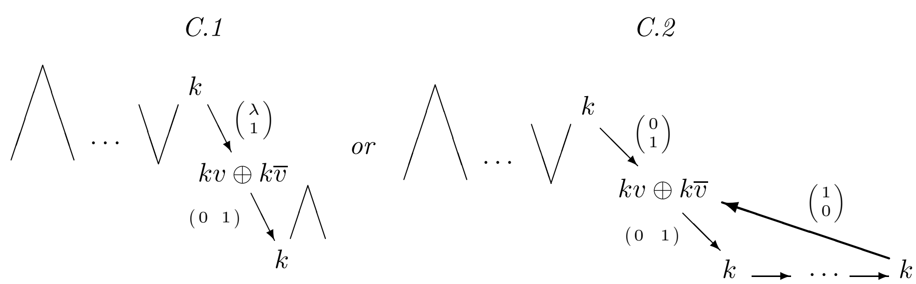
1.2: *There is in $Q_{_M}$ exactly one arrow γ with $e(\gamma) = i_{_S}$.* Since $supp\, M = supp\, \overline{M}$, $\overline{M}$ is of type of A_n and the condition (SP)2 is satisfied, it follows that $o(\gamma) = x \neq i_{_S}$ is an extreme vertex of $Q_{\overline{M}}$. Hence, since $Q_{_\Lambda}$ does not contain loops, double arrows or subquivers of type $\tilde{A}_r$ without relations (since Λ is representation-finite), it results that there is, in $Q_{\overline{M}}$, a path from i_S to x (this path is unique and x is its terminal extreme). Then there exists an oriented cycle in Q_M containing this path and γ. On the other hand, since $\gamma\colon x \to i_S$ is exactly the one arrow of Q_Λ such that $M(\gamma)\neq 0$, it follows that $k\overline{v} \cap Im\, M(\gamma) = k\overline{v} \cap \sum_\gamma Im\, M(\gamma) = (0)$, and so, by a convenient choice, we have that $M(\gamma) \cong \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, that is, M is the k-representation in B.4.

1.3: *There are, in Q_M, two different arrows γ_1 and γ_2 with $e(\gamma_i)=i_S$, for $i=1,2$.* Let $x_i = o(\gamma_i)$, for $i=1,2$. Then, $x_1 \neq x_2$ and $i_S \notin \{x_1, x_2\}$. As in 1.2, for each $i=1,2$, x_i is an extreme vertex of $Q_{\overline{M}}$ and, hence, each x_i is the terminal vertex of the unique path in $Q_{\overline{M}}$ joining i_S to x_i. So, this means that $Q_{\overline{M}}$ is the union of these paths and, consequently, Q_M is the union of the two oriented cycles which are determined by the mentioned paths above and the arrows γ_i, for $i=1,2$, having a unique common vertex, the vertex i_S. As in case 1.2, by a convenient choice, we have that $M(\gamma_i)=\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ for $i=1,2$, and, hence, M is the k-representation in B.5.

2$^{\underline{nd}}$ case: *i_S is an initial extreme of $Q_{\overline{M}}$.* Let $\delta\colon i_S \to t$ be the unique arrow such that $\overline{M}(\delta) \cong 1\!\!1$. It is clear that δ is the unique arrow of Q_M starting at i_S, and so, $M(\delta) \cong (\ 0 \quad 1\ )$. By the analysis of whether the Q_M contains arrows ending at i_S, it is easy to see, using analogous arguments, that either Q_M and $Q_{\overline{M}}$ coincide (and, in this case, M is the k-representation in B.1), or $Q_{\overline{M}}$ is a proper subpath of Q_M, obtained by elimination of one arrow ending at i_S. In this last case the quiver of M is an oriented cycle and hence M is the representation in B.2.

In both cases B.1 and B.3, it is very simple to verify that the natural inclusion of $S=kv$ in M is a splittable monomorphism. ☐

Lemma 5.3. *Let $\Lambda = k(Q_\Lambda, R_\Lambda)$ be a biserial representation-finite type algebra whose ordinary quiver Q_Λ has no loop. Let M be a k-representation whose quiver is connected. Consider $v \in M$ and the simple submodule of M, $S = kv$. Suppose that $\overline{M}=M/S$ is of type A_n, $n \geq 2$. If there is $0 \neq \overline{v} \in \overline{M}(i_S)$ such that the simple module $\overline{S} = k\overline{v}$ is not a direct summand of the module $soc\,\overline{M} \oplus top\,\overline{M}$, then M is one of the following k-representations:*



($\lambda=0$ *or* $\lambda \cong 1\!\!1$)

In case C.1, $S=kv$ is a direct summand of M.

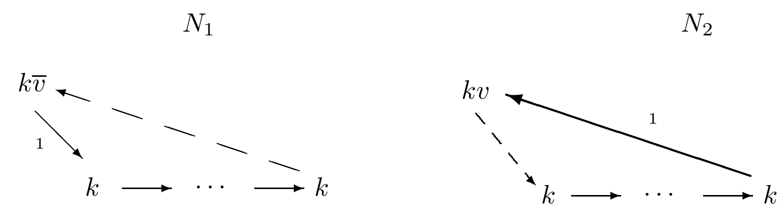
Proof. The proof of this lemma is very similar to the proof of the previous lemmas, and we leave it for the reader. ☐

Proposition 5.4. *Let $\Lambda = k(Q_\Lambda, R_\Lambda)$ be a representation-finite biserial k-algebra such that the ordinary quiver Q_Λ does not have any loop. Let M be an indecomposable Λ-module and S be a simple submodule of M such that $\overline{M}=M/S$ is of type A_n,* *with $n \geq 3$. If $\overline{M}(i_S)$ is non-zero, then M is one of the k-representations in B.2, B.4, B.5 in lemma 5.2 or C.2 in lemma 5.3.*

Proof. Let $0 \neq v \in M$ be such that $S = kv$ and $0 \neq \overline{v} \in \overline{M}(i_S)$. Consider the simple module $k\overline{v}$. Since $k\overline{v}$ is a composition factor of $\overline{M}$ and Q_M is connected (since M is indecomposable), by lemma 5.1, $k\overline{v}$ is a composition factor of $top\, M$ or is not a composition factor of $soc\, \overline{M} \oplus top\, M$. If $k\overline{v}$ is a composition factor of $top\, M$, from lemma 5.2 it follows that M is one of the representations in B.2, B.4 or B.5. If, instead, $k\overline{v}$ is not a composition factor of the module $soc\, \overline{M} \oplus top\, M$, then it is not a composition factor of the module $soc\, \overline{M} \oplus top\, \overline{M}$, and from lemma 5.3, it follows that M is the representation in C.2. □

Proposition 5.5. *Let Λ be a biserial algebra whose ordinary quiver has no loop. Suppose that the category Λ-ind is determined by composition factors. Let M be an indecomposable Λ-module, where $l(M) \geq 3$, and S be a simple Λ-module such that $S \subset M$ and $\overline{M} = M/S$ is of type A_n. Then M is multiplicity-free.*

Proof. Since Λ-ind is determined by composition factors, then by remark 1.1 Λ is a representation-finite biserial algebra and, therefore, $\Lambda \cong k(Q_\Lambda, R_\Lambda)$, under the conditions of the proposition and the conditions of remarks 4.1 and 4.2. Under the assumption on M and $\overline{M}$, we have that $M(x) \cong \overline{M}(x) \cong k$, for all $x \in supp\, M$, $x \neq i_S$. To get the result we need to prove that $\overline{M}(i_S) = 0$. Suppose that $\overline{M}(i_S) \cong k$. By proposition 5.4, M is one of the k-representations in B.2, B.4, B.5 or C.2. In any of these cases, it is immediate to see that there is a subrepresentation N_1 of $\overline{M} \cong M/kv$ and a subrepresentation N_2 of M, where $l(N_i) \geq 2$, for each $i = 1, 2$, which are represented, relative to its support, by



where the dotted arrows indicate that such arrows do not belong to the quiver of N_i, for each $i = 1, 2$ (that is, the k-morphisms to which they correspond are zero). The k-representations N_1 and N_2 are of type A_r, where $r = l(N_1) = l(N_2)$, and so, they are indecomposable modules having the same composition factors, but they are not isomorphic (since $soc\, N_2 \cong kv \ncong soc\, N_1$). We obtain a contradiction to the assumption that Λ-ind is determined by composition factors. Therefore, $M(x) \cong k$, for all vertices of the quiver of M, that is, M is multiplicity free. □

Proposition 5.6. *Let Λ be a biserial k-algebra whose ordinary quiver does not have any loop and such that Λ-ind is determined by composition factors. Let M be an indecomposable Λ-module, $l(M) \geq 3$, and S be a simple submodule of M such that $\overline{M} = M/S = N_1 \oplus N_2$, where the N_i are of type A_{n_i}, $n_i \geq 1$, for $i = 1, 2$. Then S is a composition factor of M with multiplicity one.*

Proof. As in the proof of proposition 5.5, $\Lambda \cong k(Q_\Lambda, R_\Lambda)$ satisfies the same conditions mentioned there. Let $0 \neq v \in M$ be such that $S = kv$. We consider the canonical epimorphism $\pi\colon M\to\overline{M}$ and we denote by M'_j, for $j=1,2$, the reciprocal image of N_j by π. Then, $M=M'_1+M'_2$ and $M'_1\cap M'_2=kv$. Moreover, for each $j=1,2$, $M'_j(x)=N_j(x)\cong k$, where $x\in supp\,N_j$, $x\neq i_S$, and $M(i_S)=kv\oplus N_1(i_S)\oplus N_2(i_S)$.

Suppose, for $j=1$, that $N_1(i_S)\cong k$. Then, $M'_1(i_S)=kv\oplus N_1(i_S)\cong kv\oplus k$. On the other hand, since M is an indecomposable module, by corollary 4.6, there is a unique arrow $\sigma\colon y\to i_S$ such that $v\in Im\,M'_1(\sigma)$; then $l(M_1)\geq 3$ and the quiver of M'_1 is connected. Since $M'_1/S=N_1$ is of type A_{n_1}, it follows that M'_1 is one of the representations in lemma 5.1, 5.2 or 5.3. But, since Λ-ind is determined by composition factors, from the proof of proposition 5.5, we infer that M'_1 is one of the representations in A.1, A.2, B.1 or B.3. In any of these cases, S is a direct summand of M'_1 and, consequently, a direct summand of M, which contradicts the indecomposability of M. So, $M(i_S)=kv$, that is, S has multiplicity one as a composition factor of M. □

6. The main theorems

The methods developed in the previous sections allow us now to prove the following results.

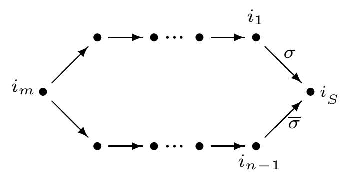
Theorem 6.1. *Let $\Lambda=k(Q_\Lambda,\,R_\Lambda)$ be a representation-finite biserial algebra. Suppose that the ordinary quiver of Λ does not have any loop and that all oriented cycles of Q_Λ are in R_Λ. If M is a multiplicity-free indecomposable Λ-module, then M either is a projective-injective non-uniserial module or is of type A_n, where $n=l(M)$.*

Proof. We first observe that $(Q_\Lambda,\,R_\Lambda)$ was chosen satisfying the conditions (SP) of theorem 4.1, and also the conditions of remark 1.1. Since M is a multiplicity-free indecomposable module, then $M(x)\cong k$, for each $x\in supp\,M$. We shall proceed by induction on the length of the multiplicity-free indecomposable modules. It is clear that each indecomposable module M, where $l(M)=n\leq 3$, is of type A_n, since it is a local or colocal module.

Suppose now that M is an indecomposable multiplicity-free module, with $l(M)=n\geq 4$, and that the result is true for all indecomposable multiplicity-free module N such that $1\leq l(N)<l(M)$. Let S be a simple submodule of M and consider the quotient M/S. If M/S is indecomposable then, by the inductive assumption, M/S is of type A_{n-1}. Let $\{i_1,i_2,...,i_{n-1}\}=supp\,M/S$ be ordered in such a way that i_1 and i_{n-1} are the extreme vertices of the quiver of M/S. Since $S\subset M$ and M is an indecomposable multipliciy-free module (therefore, Q_M is connected), then there is at least one arrow $\sigma\colon i_j\to i_S$, for some j, $1\leq j\leq n-1$ (and at most two), such that σ is in Q_M. From the conditions on $(Q_\Lambda,\,R_\Lambda)$, it follows that $o(\sigma)\in\{i_1,i_{n-1}\}$.

Suppose that the quiver of M/S is a path, for example a path from i_{n-1} to i_1. Since Q_Λ contains no subquiver of type $\tilde{A}_r$ without relations or loops, then Q_M does not contain another arrow $\overline{\sigma}\neq\sigma$ of Q_Λ ending at i_S. So, the underlying diagram of Q_M is of type A_n, and M is of type A_n, where $n=l(M)$. Otherwise, if the quiver of M/S is of type A_{n-1}, but is not a path, then there is at least one vertex i_m, $2\leq m\leq n-2$, such that i_m is either a 2-source or a 2-sink of $Q_{M/S}$.

Let us assume first that i_m is the unique 2-source of $Q_{M/S}$. Then $Q_{M/S}$ is the union of a path μ from i_m to i_1 and a path ν from i_m to i_{n-1} in $Q_{M/S}$. So, we infer that either σ is the unique arrow of Q_Λ ending at i_S which is an arrow of Q_M or, besides σ, Q_M contains another arrow $\overline{\sigma}$, where $e(\overline{\sigma})=i_S$, $o(\sigma)=i_1$ and $o(\overline{\sigma})=i_{n-1}$ such that $\sigma\mu-\overline{\sigma}\nu\in R_\Lambda$ (by (SP)4). In the first alternative, it follows imediately that the underlying graph of $Q_{_M}$ is a diagram A_n, and so, M is of type A_n. The second alternative implies that $Q_{_M}$ is



and, consequently, M is a projective-injective non-uniserial module.

Otherwise, if i_m is not the unique 2-source or i_m is a 2-sink of $Q_{_{M/S}}$, then σ is the unique arrow of $Q_{_\Lambda}$ ending at $i_{_S}$ which is an arrow of $Q_{_M}$, since $Q_{_\Lambda}$ contains no subquiver of type $\widetilde{\mathrm{A}}_r$ without relations. This implies that M is of type A_n.

Finally, we need to consider the case in which M/S is a decomposable module. By theorem 4.7, $M/S = M_1 \oplus M_2$, where, for each $i=1,2$, M_i is an indecomposable non-projective module. Moreover, each M_i is multiplicity-free and they have disjoint supports. Denoting by n_i the length of M_i, $i=1,2$, we have that M_i is of type A_{n_i} (by the induction assumption, because $l(M_i) < l(M)$, $i=1,2$). If we now consider the canonical epimorphism $\pi\colon M \to M/S$, we can write $M = N_1 + N_2$, where $N_1 \cap N_2 = S$ and N_i is the reciprocal image of M_i under π. Then, since $M(i_{_S}) \cong k$ and $supp\, N_1 \cap supp\, N_2 = \{i_{_S}\}$, by corollary 4.6, there is a unique arrow $\sigma_1\colon y_1 \to i_{_S}$ such that $N_1(\sigma_1) \cong \mathbb{1}$ and there is a unique arrow $\sigma_2\colon y_2 \to i_{_S}$, where $\sigma_2 \neq \sigma_1$ and $y_1 \notin \{y_2, i_{_S}\}$, such that $N_2(\sigma_2) \cong \mathbb{1}$. Moreover, for each $i=1,2$, $y_i \in supp\, M_i$ is a extreme vertex of the quiver $Q_{_{M_i}}$. So, we conclude that $Q_{_M}$ has for its underlying graph a diagram A_n, and the theorem is proved. ☐

Theorem 6.2. *Let Λ be a connected, basic biserial k-algebra whose ordinary quiver does not have a loop. Suppose that Λ-ind is determined by composition factors. Then each indecomposable Λ-module is multiplicity-free.*

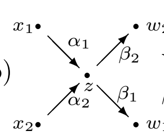
Proof. Since Λ-ind is determined by composition factors, by remark 1.1, Λ is a representation-finite biserial algebra, and so, we choose $\Lambda \cong k(Q_{_\Lambda}, R_{_\Lambda})$, where $(Q_{_\Lambda}, R_{_\Lambda})$ satisfies the conditions (SP) of theorem 4.1 and those of proposition 3.1. Moreover, according to remark 4.2, all oriented cycles of $kQ_{_\Lambda}$ are in $R_{_\Lambda}$. We shall proceed by induction on the length of the indecomposable modules. Let M be in Λ-ind and such that $l(M) \leq 3$. Is easy to verify that each indecomposable of length less than or equal to 3 is a local or colocal module, and, by theorem 4.3, it follows that M is multiplicity-free. Suppose that M is indecomposable such that $l(M) \geq 4$ and that the result is true for each indecomposable module N such that $1 \leq l(N) < l(M)$. By duality it is enough to prove that if S is a composition factor of M which is a direct summand of $soc\, M$ or S is not a composition factor of $soc\, M \oplus top\, M$, then the multiplicity of S in M is one.

Let S be a simple submodule of M and consider the quotient M/S. If M/S is indecomposable, by the induction assumption it follows that M/S is multiplicity-free. Since M/S is not a projective module (since M is indecomposable), then, by theorem 6.1, it is of type A_{n-1}, where $n = l(M)$, and so, by proposition 5.5, the multiciplity of S in M is one. If M/S is not indecomposable, then, by theorem 4.7, it follows that $M/S = M_1 \oplus M_2$, where each $M_i \neq (0)$ is an indecomposable non-projective module, for $i=1,2$. Again by the induction assumption and theorem 6.1, each M_i is of type A_{n_i}, where $n_i = l(M_i)$. Hence, by proposition 5.6, the multiplicity of S in M is one. So, we conclude that each simple submodule of M has multiplicity one as a composition factor of M.

Let T be a simple module which is a composition factor of M but not a composition factor of the module $soc\,M \oplus top\,M$. Suppose that the multiplicity of T in M is greater than or equal to 2, that is, $dim_k M(i_T) \geq 2$. Let S be a simple submodule of M. By the proof above we have that $dim_k M(i_S) = 1$, and so $S \ncong T$. Consider the quotient M/S. Since $dim_k(M/S)(i_T) \geq 2$, by the induction assumption (for $l(M/S) < l(M)$) and by theorem 4.7, it follows that $M/S = M_1 \oplus M_2$, where $M_i \neq (0)$, $i = 1, 2$, are indecomposable non-projective modules, whose $dim_k M_i(i_T) = 1$, for each $i = 1, 2$, and $dim_k M(i_T) = 2$. Moreover, since $l(M_i) < l(M)$, for each $i = 1, 2$, by the induction assumption and theorem 6.1, each M_i is of type A_{n_i}, where $n_i = l(M_i) \geq 1$. Now let $\pi\colon M \to M/S$ be the canonical epimorphism, and let the submodules $N_i = \pi^{-1} M_i$, $i = 1, 2$, be such that $M = N_1 + N_2$, $N_1 \cap N_2 = S = kv$, where $v \in M(i_S)$, and $N_i/S = M_i$, for $i = 1, 2$. From corollary 4.6, it follows that there are arrows $\sigma_1\colon y_1 \to i_S$ and $\sigma_2\colon y_2 \to i_S$, where $\sigma_1 \neq \sigma_2$ (hence, $y_1 \neq i_S$ and $y_2 \notin \{y_1, i_S\}$), such that $Im\, N_i(\sigma_i) = kv$, for each $i = 1, 2$ and $N_i(\sigma_j) = 0$ if $i \neq j$. Then, each N_i is of type A_{n_i+1} and i_S is a terminal extreme of the quiver Q_{N_i}, for $i = 1, 2$. Hence, i_S is a 2-sink of Q_M and $\{i_S, i_T\} \subset supp\,N_1 \cap supp\,N_2$. We claim that if $z \in supp\,N_1 \cap supp\,N_2$, $z \neq i_S$, then z is a transit point of Q_{N_1} and of Q_{N_2}. In fact, since $soc\,N_i \subset soc\,M$, for each $i = 1, 2$, and each composition factor of $soc\,M$ has multiplicity one in M, then the simple module S_z is not a composition factor of $soc\,N_i$, for all $i = 1, 2$. So, z is not a sink of Q_{N_1}, nor of Q_{N_2}. On the other hand, since $top\,M = top\,N_1 \oplus top\,N_2$ and each composition factor of $top\,M$ has multiplicity one in M (for duality of the $soc\,M$), then, for each $i = 1, 2$, no direct summand of $top\,N_i$ is isomorphic to S_z. Therefore, z is not a source of Q_{N_1}, nor of Q_{N_2}. Our claim is proved.

Therefore, the set $supp\,N_1 \cap supp\,N_2$ contains the vertices of M which are transit point of Q_{N_1} and of Q_{N_2}. So, the neighbourhood of these vertices $z \neq i_S$ in each of these subquivers has the following figure: $x_i \xrightarrow{\alpha_i} z \xrightarrow{\beta_i} w_i$ such that $\beta_i \alpha_i \notin R_\Lambda(x_i, w_i)$, for each $i = 1, 2$. From the condition (SP)2, it follows that if $\alpha_1 = \alpha_2$ then $\beta_1 = \beta_2$, and if $\alpha_1 \neq \alpha_2$, then $\beta_1 \neq \beta_2$, $\beta_2 \alpha_1 \in R_\Lambda(x_1, w_2)$ and $\beta_1 \alpha_2 \in R_\Lambda(x_2, w_1)$. Thus the neighbourhood of each $z \in supp\,N_1 \cap supp\,N_2$, with $z \neq i_S$, is one of the following figures:

(a) $\underset{x_1}{\bullet} \xrightarrow{\alpha_1} \underset{z}{\bullet} \xrightarrow{\beta_1} \underset{w_1}{\bullet}$ with $\beta_1 \alpha_1 \notin R_\Lambda$ or (b)

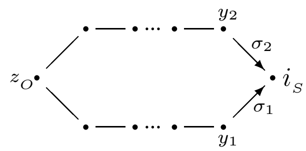


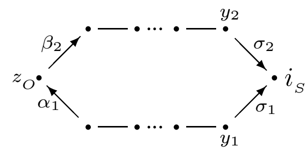
where $\beta_i \alpha_i \notin R_\Lambda$, $i = 1, 2$, $\beta_2 \alpha_1 \in R_\Lambda$ and $\beta_1 \alpha_2 \in R_\Lambda$.

We observe that in the case (a) we have that $\{x_1, w_1\} \subset supp\,N_1 \cap supp\,N_2$. For each $z \in supp\,N_1 \cap supp\,N_2$, where $z \neq i_S$, and for each $i = 1, 2$, let $A_i(i_S, z)$ be the walk of minimal length of the quiver N_i joining i_S to z. Then, for each i, $A_i(i_S, z)$ contains the arrow σ_i and is, in particular, a subquiver of type A_{r_i}, where $r_i = l(A_i(i_S, z)) + 1$ and $l(A_i(i_S, z))$ denotes the number of arrows of $A_i(i_S, z)$. Now, we choose, for each $i = 1, 2$, $z_i \in supp\,N_1 \cap supp\,N_2$, where $z_i \neq i_S$, such that $l(A_i(i_S, z_i))$ is the least among the vertices $z \neq i_S$ which are in $supp\,N_1 \cap supp\,N_2$. This choice implies that we need to annalyse two cases: $z_1 = z_2$ or $z_1 \neq z_2$.

$1^{\underline{st}}$ case: $z_1 = z_2 = z_0$**.** The minimality of the length of $A_i(i_S, z_0)$, $i = 1, 2$, implies that the neighborhood of z_0 in Q_M is as in figure (b) and that i_S and

z_0 are the unique common vertices of $A_1(i_S, z_0)$ and $A_2(i_S, z_0)$. Then Q_M and, consequently, Q_Λ, contain one of the following subquivers, which are the union of the walks $A_i(i_S, z_0)$, $i=1,2$:

(c) 

(d) 

where in (c) z_0 is a 2-source or a 2-sink.

In (c), since Q_Λ contains no subquivers of type $\tilde{A}_r$ without relations, it is easy to see that each $A_i(i_S, z_0)$, $i=1,2$, is a path from z_0 to i_S. So, they are parallel and such that $A_1(i_S, z_0)-A_2(i_S, z_0)\notin R_\Lambda$ (for $A_i(i_S, z_0)\alpha_i \notin R_\Lambda$), which is a contradiction to condition (SP)4 of theorem 4.1. If we have case (d), we construct the families $N'=(N'(x)_x, N'(\delta)_\delta)$ and $N''=(N''(x)_x, N''(\delta)_\delta)$, where x denotes the vertices of Q_Λ and δ their arrows, defined by:

$$N'(x)=\begin{cases} N_2(x) & \text{if } x \text{ is in } A_2(i_S, z_0),\ x\neq i_S,\\ N_1(x) & \text{if } x \text{ is in } A_1(i_S, z_0),\ x\notin\{z_0,\ i_S\},\\ kv & \text{if } x=i_S,\\ (0) & \text{for the other vertices,}\end{cases}$$

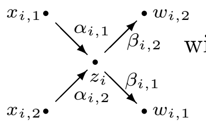
$$N'(\delta)=\begin{cases} N_2(\delta) & \text{if } \delta \text{ is in } A_2(i_S, z_0),\\ N_1(\delta) & \text{if } \delta \text{ is in } A_1(i_S, z_0),\ \delta\neq\alpha_1,\\ 0 & \text{for the other arrows,}\end{cases}$$

$$N''(x)=\begin{cases} N_2(x) & \text{if } x \text{ is in } A_2(i_S, z_0),\ x\notin\{z_0,\ i_S\},\\ N_1(x) & \text{if } x \text{ is in } A_1(i_S, z_0),\ x\neq i_S,\\ kv & \text{if } x=i_S,\\ (0) & \text{for the other vertices,}\end{cases}$$

$$N''(\delta)=\begin{cases} N_2(\delta) & \text{if } \delta \text{ is in } A_2(i_S, z_0),\ \delta\neq\beta_2,\\ N_1(\delta) & \text{if } \delta \text{ is in } A_1(i_S, z_0),\\ 0 & \text{for the other arrows.}\end{cases}$$

By the definition of N' and N'', it is easy to verify that they are k-representations such that *supp* N' = *supp* N''. Moreover, the quivers $Q_{N'}$ and $Q_{N''}$ are obtained from the quiver in (d) by deleting, respectively, the arrows α_1 and β_2; and it follows that N' (N'') is a module of type A_r, where $r=l(A_1(i_S, z_0))+l(A_2(i_S, z_0))$, whose *top* (respectively, whose *soc*) contains a direct summand isomorphic to the simple module S_{z_0}, since the vertex z_0 is a initial extreme of $Q_{N'}$ (respectively, since z_0 is a terminal extreme of $Q_{N''}$). So we construct two multiplicity-free modules (N' and N'') having the same composition factors, but non-isomorphic, which is a contradiction.

2$^{\underline{nd}}$ case: $z_1 \neq z_2$. From the minimality of the length of $A_1(i_S, z_1)$ and of $A_2(i_S, z_2)$, it follows that i_S is the unique common vertex of them and, analogously to the first case, the neighbourhood of z_1 and of z_2 in Q_M are as in figure (b). So, we indicate their arrows as well as the vertices of the neighbourhood of each z_i, using double indices, in the following way: the first index i corresponds to the vertex z_i and the second one j to which Q_{N_j} belongs, as the following figure shows:

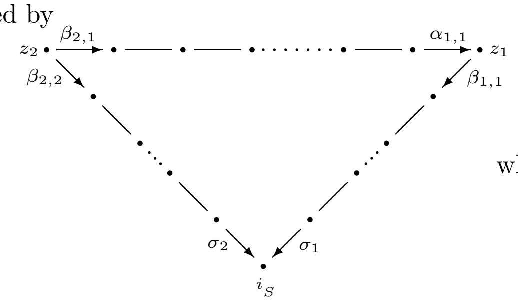


with $\beta_{i,1}\alpha_{i,1} \notin R_{\Lambda}$, $\beta_{i,2}\alpha_{i,2} \notin R_{\Lambda}$, $\beta_{i,2}\alpha_{i,1} \in R_{\Lambda}$ and $\beta_{i,1}\alpha_{i,2} \in R_{\Lambda}$, for $i=1,2$.

Since, for each $i=1,2$, Q_{N_i} is connected, we consider the walk $A_1(z_1,z_2)$ in Q_{N_1} and the walk $A_2(z_1,z_2)$ in Q_{N_2}, such that they are the walks of the least length, in each corresponding quiver, joining z_1 to z_2. It is clear, by the assumption on minimality of the length, that z_1 and z_2 are their unique common vertices. So, we have many different possibilities, according to whether the arrows $\alpha_{i,j}$ and $\beta_{i,j}$, for i, $j=1,2$, belong (or not) to the walks $A_i(i_S,z_i)$ and $A_i(z_1,z_2)$, which are the following.

2.1: *For all $i=1,2$, $A_i(i_S,z_i)$ contains the arrow $\beta_{i,i}$.* Under this condition, each walk $A_i(z_1,z_2)$ contains the arrow $\alpha_{i,i}$ and exactly one of $\alpha_{i,j}$ or $\beta_{i,j}$ if $i \neq j$. Therefore, there are two subcases.

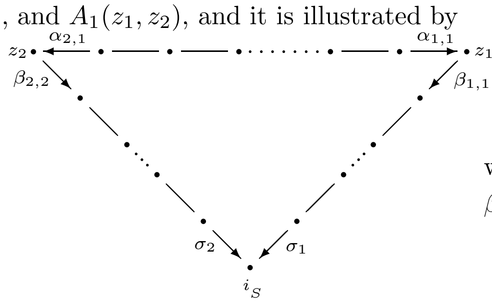
2.1.1: *For some i, for example $i=1$, $A_1(z_1,z_2)$ contains the arrow $\beta_{2,1}$.* Then the union of the walks $A_i(i_S,z_i)$, for $i=1,2$ and $A_1(z_1,z_2)$ determines a subquiver of type $\tilde{\mathrm{A}}_r$ without relations, where r is the sum of the lengths of those walks, which is illustrated by



where $\beta_{1,1}\alpha_{1,1} \notin R_{\Lambda}$.

This contradicts the fact that Λ is a representation-finite algebra.

2.1.2: *For some i, for example $i=1$, $A_1(z_1,z_2)$ contains the arrow $\alpha_{2,1}$.* Under this condition, Q_M contains a subquiver which is the union of the walks $A_i(i_S,z_i)$, $i=1,2$, and $A_1(z_1,z_2)$, and it is illustrated by



where $\beta_{1,1}\alpha_{1,1} \notin R_{\Lambda}$ and $\beta_{2,2}\alpha_{2,1} \in R_{\Lambda}$.

In this subcase, let $N'=(N'(x)_x, N'(\alpha)_\alpha)$ and $N''=(N''(x)_x, N''(\alpha)_\alpha)$ be the families defined by

$$N'(x)=\begin{cases} N_2(x) & \text{if } x \in A_2(i_S,z_2),\ x \notin \{z_2, i_S\}, \\ N_1(x) & \text{if } x \in A_1(i_S,z_1) \cup A_1(z_1,z_2) \text{ and } x \neq i_S, \\ kv & \text{if } x = i_S, \\ (0) & \text{for the other vertices,} \end{cases}$$

$$N'(\alpha) = \begin{cases} N_2(\alpha) & \text{if } \alpha \in A_2(i_S, z_2) \text{ and } \alpha \neq \beta_{2,2}, \\ N_1(\alpha) & \text{if } \alpha \in A_1(i_S, z_1) \cup A_1(z_1, z_2), \\ 0 & \text{for the other arrows,} \end{cases}$$

$$N''(x) = \begin{cases} N_2(x) & \text{if } x \in A_2(i_S, z_2) \text{ and } x \neq i_S, \\ N_1(x) & \text{if } x \in A_1(i_S, z_1) \cup A_1(z_1, z_2) \text{ and } x \notin \{z_2, i_S\}, \\ kv & \text{if } x = i_S, \\ (0) & \text{for the other vertices,} \end{cases}$$

$$N''(\alpha) = \begin{cases} N_2(\alpha) & \text{if } \alpha \in A_2(i_S, z_2), \\ N_1(\alpha) & \text{if } \alpha \in A_1(i_S, z_1) \cup A_1(z_1, z_2), \\ & \text{and } \alpha \neq \alpha_{2,1}, \\ 0 & \text{for the other arrows.} \end{cases}$$

It is easy to verify that N' and N'' are k-representations whose quivers are obtained from the quiver above by deleting the arrows $\beta_{2,2}$ and $\alpha_{2,1}$, respectively. Therefore, using the same arguments used in 2.1.2, we conclude that N' and N'' are indecomposable multiplicity-free modules having the same composition factors (since they have the same support), but they are not isomorphic (since $S_{z_2} \subset soc\, N'$ and $S_{z_2} \not\subset soc\, N''$). Again we obtain a contradiction to the assumption on Λ-ind.

2.2: *For some i, for example $i = 1$, $A_1(i_S, z_1)$ contains the arrow $\alpha_{1,1}$.* Then, $A_1(z_1, z_2)$ necessarily contains the arrow $\beta_{1,1}$. Relative to the other arrows to which z_1 and z_2 belong, we have the following subcases.

2.2.1: *$A_1(z_1, z_2)$ contains the arrow $\beta_{2,1}$ (or $\alpha_{2,1}$) and $A_2(i_S, z_2)$ contains the arrow $\beta_{2,2}$ (respectively, $\alpha_{2,2}$)*;

2.2.2: *$A_1(z_1, z_2)$ contains $\beta_{2,1}$ and $A_2(i_S, z_2)$ contains $\alpha_{2,2}$*; and

2.2.3: *$A_1(z_1, z_2)$ contains $\alpha_{2,1}$ and $A_2(i_S, z_2)$ contains $\beta_{2,2}$*. Since subcase 2.2.1 is similar to subcase 2.1.1 and the others are similar to 2.1.2, we leave it to the reader to verify that they lead to a contradiction to the assumptions. Hence, we cannot suppose that T does not have multiplicity one as a composition factor of M. So it is proved that each indecomposable module is multiplicity-free. □

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